# EXPANSIONS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH RETARDATION IN POWER SERIES OF SMALL RETARDATION

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**1.** In [1], the equation with retarded argument

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-\tau)), \qquad x(t) = \varphi(t) \quad \text{on } E_0 \tag{1.1}$$

has been treated by the perturbation method, in seeking to determine an asymptotic expansion of the solution $x(t, \tau)$ in terms of a power series in small retardation $\tau$

$$x(t, \tau) = x_0(t) + \tau x_1(t) + \frac{\tau^2}{2} x_2(t) + \ldots \tag{1.2}$$

In the case of equation

$$\frac{dx(t)}{dt} = ax(t-\tau), \qquad x(t) = \varphi(t) \quad \text{on } E_0 \tag{1.3}$$

whose solution may be obtained by a step by step method, the expansion (1.2) coincides with the expansion of the solution $x(t, \tau)$ of (1.3) by Taylor's formula. In this connection, the question arises as to the differentiability of the solutions of Equation (1.1) with respect to $\tau$. In this note the following theorem will be proved:

*Theorem.* If the function $f(t, x(t), x(t-\tau))$ has continuous partial derivatives, with respect to all its arguments, up to order $N$, and the function $\varphi(t)$ has continuous derivatives with respect to $t$ up to order $N$, then, for each fixed $t^* > 0$ and a sufficiently large $N$, there always exists a number $\tau^*(t^*, N) = t^*/N$, such that the function $x(t^*, \tau)$ has $N$ continuous derivatives with respect to $\tau$ on the interval $0 \leqslant \tau < \tau^*$

$$\partial x(t^*, \tau)/\partial\tau, \ldots, \partial^N x(t^*, \tau)/\partial\tau^N$$

From this follows the possibility of expanding, for sufficiently small $\tau$, the function $x(t, \tau)$ using Taylor's formula

$$x(t, \tau) = x(t, 0) + \tau x'_\tau(t, 0) + \frac{\tau^2}{2} x''_\tau(t, 0) + \dots$$

*Proof.* Suppose that the initial condition in (1.1) is of the form

$$x(t) = \varphi(t) \quad \text{for } -\infty < t \leqslant 0 \tag{1.4}$$

Equations (1.1) and (1.4) determine the solution $x(t, \tau)$, as a function of $t$ and $\tau$, on the interval $0 \leqslant t \leqslant T < +\infty$. Let the variable parameter $\tau$ be $0 \leqslant \tau < \tau_1$. Suppose that the retardation is chosen from the number interval just mentioned. Substituting $x(t, \tau)$ in (1.1), we obtain the identity

$$\frac{dx(t, \tau)}{dt} = f(t, x(t, \tau), x(t-\tau, \tau)) \tag{1.5}$$

For an arbitrary, sufficiently small, increment $\Delta\tau$ of $\tau$, we obtain the corresponding identity

$$\frac{dX(t, \tau+\Delta\tau)}{dt} = f(t, X(t, \tau+\Delta\tau), \qquad X(t-(\tau+\Delta\tau), \tau+\Delta\tau) \tag{1.6}$$

Let us subtract (1.5) from (1.6), and apply the theorem of the mean of the differential calculus, to obtain

$$\frac{d(X-x)}{dt} = f(t, X, X_\tau) - f(t, x, x_\tau) = f(t, X, X_\tau) -$$
$$- f(t, x, X_\tau) + f(t, x, X_\tau) - f(t, x, x_\tau) =$$
$$= \frac{\partial f(t, x+\theta_1(X-x), X_\tau)}{\partial x}(X-x) + \frac{\partial f(t, x, x_\tau+\theta_2(X_\tau - x_\tau))}{\partial x_\tau}(X_\tau - x_\tau) \tag{1.7}$$
$$(0 < \theta_1, \ \theta_2 < 1)$$

Transforming the difference $X_\tau - x_\tau$ by means of the theorem of the mean, we obtain

$$X_\tau - x_\tau = x(t-(\tau+\Delta\tau), \tau+\Delta\tau) - x(t-\tau, \tau) =$$
$$= x(t-(\tau+\Delta\tau), \tau+\Delta\tau) - x(t-\tau, \tau+\Delta\tau) + x(t-\tau, \tau+\Delta\tau) - x(t-\tau, \tau) =$$
$$= -\frac{dx(t-\tau-\theta_3\Delta\tau, \tau+\Delta\tau)}{dt}\Delta\tau + x(t-\tau, \tau+\Delta\tau) - x(t-\tau, \tau)$$
$$(0 < \theta_3 < 1)$$

while, upon dividing both sides of Equation (1.7) by $\Delta\tau$, the result is

$$\frac{d}{dt}\frac{X-x}{\Delta\tau} = \frac{\partial f(t, x+\theta_1(X-x), X_\tau)}{\partial x}\left(\frac{X-x}{\Delta\tau}\right) +$$
$$+ \frac{\partial f(t, x, x_\tau+\theta_2(X_\tau - x_\tau))}{\partial x_\tau}\left\{\left(\frac{X-x}{\Delta\tau}\right)_\tau - \frac{dx(t-\tau-\theta_3\Delta\tau, \tau+\Delta\tau)}{dt}\right\} \tag{1.8}$$

In a certain neighborhood of zero: $0 \leqslant t \leqslant h < \tau$, the derivatives

$\partial f(\ldots)/\partial x$, $\partial f(\ldots)/\partial \tau$, and $dx(\ldots)/dt$ are continuous functions of the two variables $t$ and $\tau$, jointly. Indeed, if $X - x \neq 0$, $X_\tau - x_\tau \neq 0$, $\Delta\tau \neq 0$, then the asserted continuity of these derivatives follows from the equations:

$$\frac{\partial f(\ldots)}{\partial x} = \frac{f(t, X, X_\tau) - f(t, x, X_\tau)}{X - x} \qquad \frac{\partial f(\ldots)}{\partial x_\tau} = \frac{f(t, x, X_\tau) - f(t, x, x_\tau)}{X_\tau - x_\tau}$$

$$\frac{dx(\ldots)}{dt} = \frac{x(t - \tau, \tau + \Delta\tau) - x(t - (\tau + \Delta\tau), \tau + \Delta\tau)}{\Delta\tau}$$

In these equations, both the numerator and the denominators are continuous (see [2]), and the denominator is not equal to zero. Since, as $t \to \bar{t}$, $\Delta\tau \to \overline{\Delta\tau}$, one has that $X - x \to 0$, $X_\tau - x_\tau \to 0$, and $\Delta\tau \to 0$; then, in view of the continuity of the derivatives in question with respect to the totality of the arguments, the ratios in question tend, respectively, to the limits

$$\frac{\partial f(t, x(t), x(t-\tau))}{\partial x}, \qquad \frac{\partial f(t, x(t), x(t-\tau))}{\partial x_\tau}, \qquad \frac{dx(t-\tau, \tau)}{dt}.$$

Equation (1.8) may be regarded as a linear nonhomogeneous equation with retardation $\tau$ and unknown function $X - x/\Delta\tau$. Its coefficients, by what has been demonstrated, are continuous functions of the two variables $t$ and $\tau$ jointly (for sufficiently small $|\Delta\tau|$). The initial function for the sought solution, obviously, is $\varphi(t) \equiv 0$ on $E_0$. Consequently, applying to Equation (1.8) the theorem on the continuous dependence of solutions on the right-hand side [2], we obtain that the solution of Equation (1.8) also depends continuously on $\Delta\tau$; in particular, the following limit exists:

$$\lim_{\Delta\tau\to 0} \frac{X - x}{\Delta\tau} = \lim_{\Delta\tau\to 0} \frac{x(t, \tau + \Delta\tau) - x(t, \tau)}{\Delta\tau} = \frac{\partial x(t, \tau)}{\partial\tau}$$

Thus, for $0 \leqslant t \leqslant h$ there exists a continuous derivative $\partial x(t, \tau)/\partial\tau$, which satisfies the linear nonhomogeneous equation

$$\frac{dz(t)}{dt} = \frac{\partial f(t, x(t), x(t-\tau))}{\partial x} z(t) + \frac{\partial f(t, x(t), x(t-\tau))}{\partial x_\tau} z(t-\tau) - \\ - \frac{\partial f(t, x(t), x(t-\tau))}{\partial x_\tau} \frac{dx(t-\tau, \tau)}{dt} \tag{1.9}$$

As is well known [3], the derivative $dx(t)/dt$ of the solution, generally speaking, is discontinuous at $t = 0$. Hence $z(t) = \partial x(t, \tau)/\partial\tau$, generally speaking, is not defined at the point $t = \tau$. In the coordinate plane $(t, \tau)$, the equation $t = \tau$ defines a "line of discontinuity" of the derivative $\partial x(t, \tau)/\partial\tau$. For $t > \tau$ the derivative $dx(t - \tau, \tau)/dt$ of the solution, and together with it also $\partial x(t, \tau)/\partial\tau$, are continuous.

Equation (1.9) we shall call the equation of variations. The equation of variations for the second derivative $\partial^2 x(t, \tau)/\partial\tau^2$ will contain $d^2 x(t - \tau, \tau)/dt^2$, that is, $\partial^2 x(t, \tau)/\partial\tau^2$ will, in general, be not defined at the point $t = 2\tau$. In the coordinate plane $(t, \tau)$, the equation $t = 2\tau$ defines a line of discontinuity of the second derivative of the solution with respect to $\tau$, that is $\partial^2 x(t, \tau)/\partial\tau^2$. Analogously, we may construct the equations of variations for third, ..., and $n$th derivatives, and arrive at the conclusion that the lines $t = n\tau$ (for $n = 1, 2, \ldots$) are lines of discontinuity for the derivatives $\partial^n x(t, \tau)/\partial\tau^n$. Let us construct the lines $t = n\tau$ (for $n = 1, 2, \ldots$) in the $t\tau$ plane.

From the figure there follows the asserted conclusion concerning the differentiability of the solution of Equation (1.1) with respect to $\tau$, for a fixed value of $t^*$.

**2.** The method of expanding solutions in series of a small retardation parameter may be employed in calculating periodic solutions of systems with retardation

$$\frac{dx(t)}{dt} = f(t, x(t), x(t - \tau)) \tag{2.1}$$

where $x(t) = \{x_1(t), \ldots, x_n(t)\}$ is a vector function, and the function $f$ has period $2\pi$ in $t$.

Let us suppose that the system (2.1) possesses a unique periodic solution $x(t)$, with period $2\pi$, for $0 \leqslant \tau < \varepsilon$; which, as is well known, is defined by an initial function which is a periodic extension of $x(t)$ to the initial set. It is clear [3] that the periodic solution $x(t)$ will be infinitely differentiable if the function $f(t, x(t), x(t - \tau))$ is also infinitely differentiable. Consequently, by the methods of [1] the solution $x(t)$ may be approximated to any desired degree of accuracy

$$x(t) = x_0(t) + \tau x_1(t) + \ldots + \frac{\tau^n}{n!} x_n(t) + O(\tau^{n+1}) \quad (0 \leqslant t \leqslant 2\pi)$$

where $x_0(t)$ is a solution of the system

$$\frac{dx_0(t)}{dt} = f(t, x_0(t), x_0(t)) \tag{2.2}$$

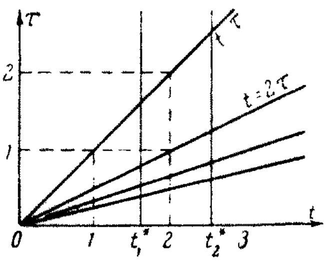
and $x_n(t)$ (for $n = 1, 2, \ldots$) is a solution of the system

$$\frac{dx_n(t)}{dt} = \left[\frac{\partial f(t, x_0(t), x_0(t))}{\partial x} + \frac{\partial f(t, x_0(t), x_0(t))}{\partial x_\tau}\right] x_n(t) + f_n(t) \qquad (n = 1, 2, \ldots) \tag{2.3}$$

where $f_n(t)$ depends on $x_0(t)$, $x_1(t)$, ..., $x_{n-1}(t)$ and is a known function of time. In the case under consideration Equation (2.2) has a unique periodic solution with period $2\pi$. Thus (2.3) is a linear nonhomogeneous system with periodic coefficients of period 2, and with a known periodic function $f_n(t)$. Let us suppose that the homogeneous system corresponding to (2.3) possesses only the trivial periodic solution. Then (2.2) and (2.3) define a unique periodic solution

$$x_0(t) + \tau x_1(t) + \ldots + \frac{\tau^n}{n!} x_n(t)$$

which approximates the periodic solution $x(t)$ of the system (2.1) up to terms of order $\tau^{n+1}$.




By way of an example, consider the equation

$$\dot{x}(t) + ax(t) + b(t)x(t-\tau) = f(t) \qquad (a > 0) \quad (2.4)$$

where $\tau$ is a small retardation, and $b(t)$ and $f(t)$ are periodic functions with period $2\pi$. Let us assume that $|b(t)| < a$. Then, for the equation

$$\dot{x}(t) + ax(t) + b(t)x(t-\tau) = 0 \qquad (2.5)$$

one may construct a functional which satisfies the hypotheses of a theorem of Krasovskii [5] concerning uniform asymptotic stability. Consequently, there exists a unique periodic solution of Equation (2.5), namely $x(t) \equiv 0$. According to [4], Equation (2.4) has a unique periodic solution $x(t)$, which may be approximated to any desired degree of accuracy by means of the method outlined above. Let us remark that, for Equation (2.4), the results obtained in this way have something in common with the results of Krasovskii [5].

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